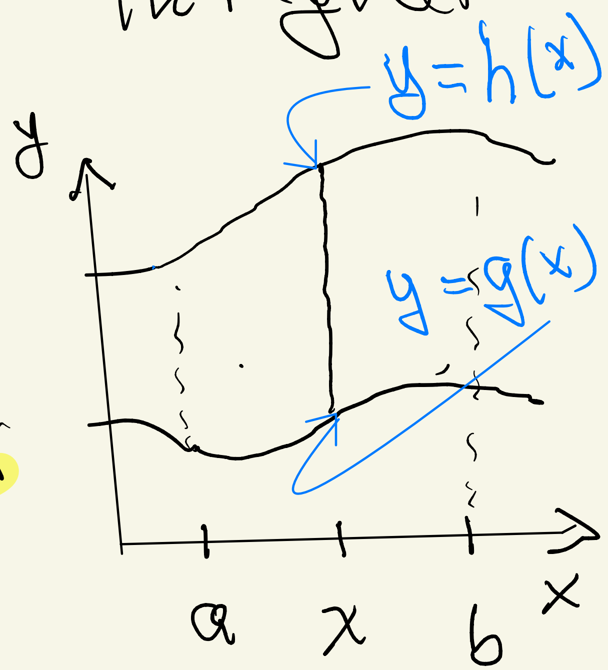
§15.3 Polar Coordinates

- Recall the defn of integral:

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{(x_i,y_j)\in R} f(x_i,y_j)\Delta x \Delta y}_{\text{2-D Riemann Sum}}$$

- To evaluate: iterate the integral

$$\iint_R f(x,y)\,dA = \int_a^b \int_{g(x)}^{h(x)} f(x,y)\,dy\,dx$$



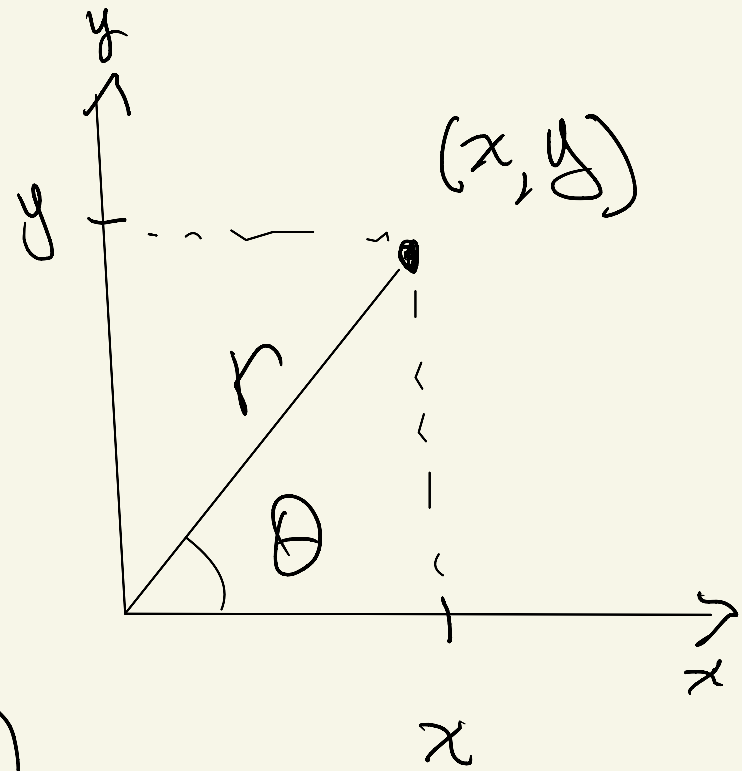
- when the function f is radially symmetric the integral can be evaluated more easily in polar coordinates
- The idea: change variables from $(x, y) \longrightarrow (r, \theta)$, and then write the Riemann Sum in terms of r and θ . . .

Ie $$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \underbrace{\sum_{R_{r\theta}}}_{\text{Sum in } (r,\theta)} \underbrace{\tilde{f}(r_i,\theta_j)\,\Delta r\,d\theta}_{\text{Riemann}}$$

- Recall the expression for x and y in terms of r and θ:

$x = r\cos\theta$

$y = r\sin\theta$



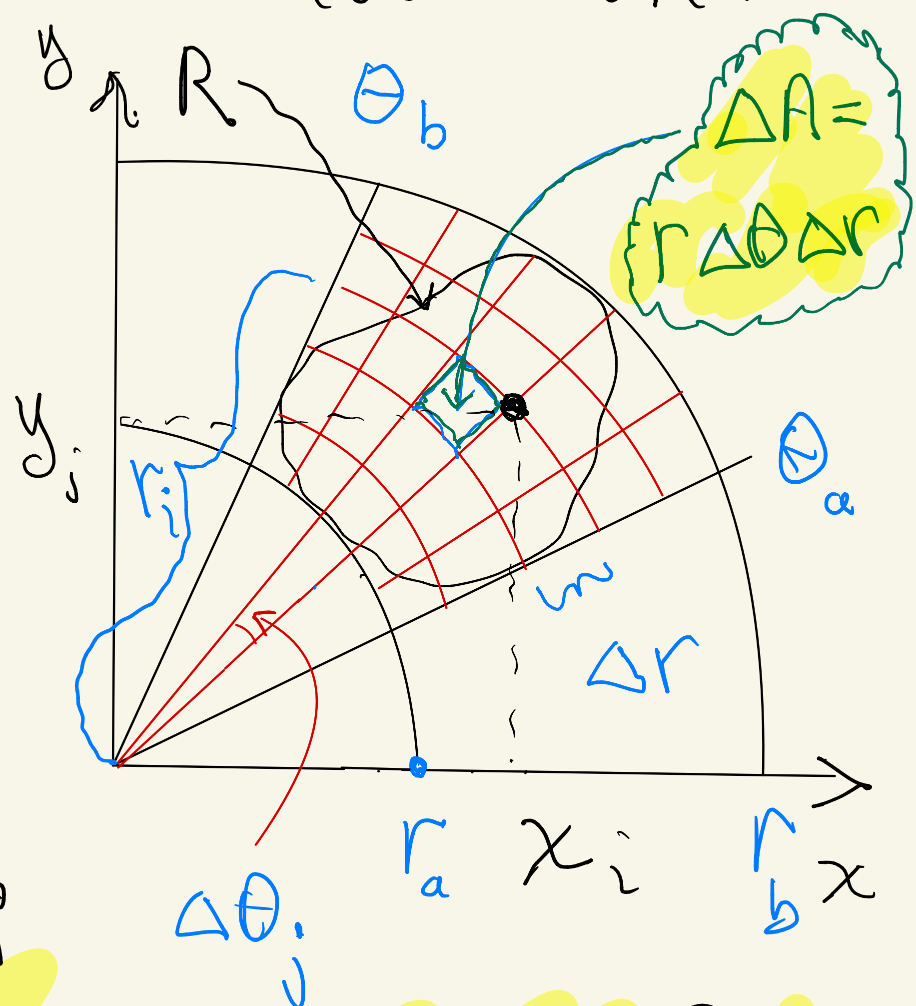
- Thus its easy to write $f(x,y)$ in terms of r & θ:

$$f(x,y) = f(r\cos\theta, r\sin\theta) = \bar{f}(r,\theta)$$

- Q: How does the area change betw $\Delta x \Delta y$ & $\Delta r \Delta\theta$?

• So consider the problem of evaluating $\iint_R f(x,y)\,dA$ in polar coordinates –

To do this we write integral as a Riemann Sum in (r,θ)



$$x_i = r_i\cos\theta_j$$

$$y_j = r_i\cos\theta_j$$

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \overbrace{\sum_{(r_i,\theta_j)\in R_{r\theta}} \tilde{f}(r_i,\theta_j)\,\Delta r\,\Delta\theta}^{\text{Riemann Sum in } r,\theta}$$

5

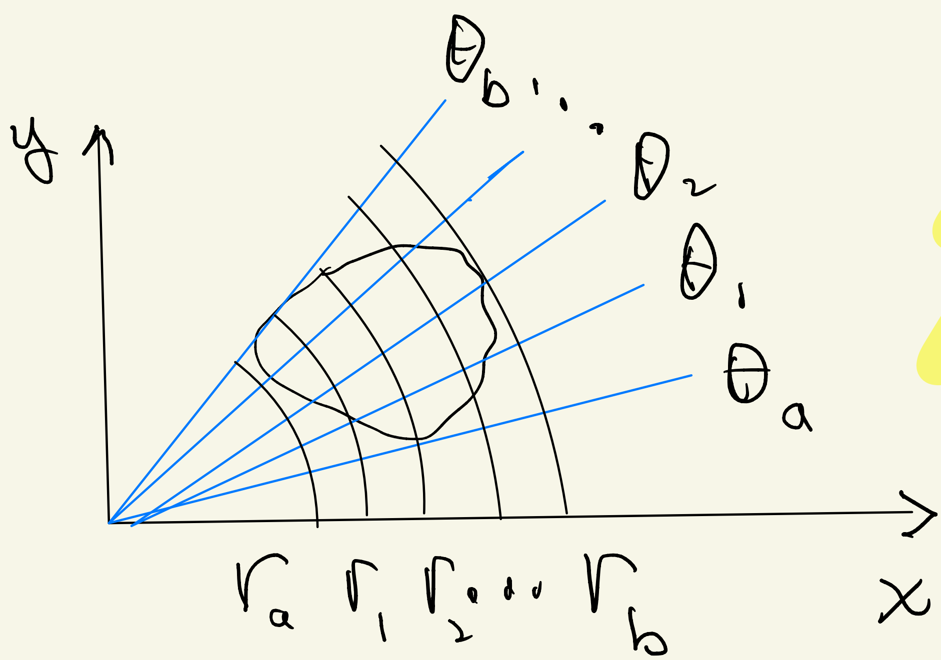
• That is: Draw the region $R_{r\theta}$ in xy-coordinates, cover it with a grid

$$r_a = r_0 < r_1 < \cdots < r_N = r_b$$

$$\theta_a = \theta_0 < \theta_1 < \cdots < \theta_N = \theta_b$$

$$\Delta r = \frac{r_b - r_a}{N}, \quad r_i = r_a + i\,\Delta r$$

$$\Delta\theta = \frac{\theta_b - \theta_a}{N}, \quad \theta_j = \theta_a + j\,\Delta\theta$$



View $R_{r\theta}$ in the xy-plane!

- **Main question**: what is the amplification factor? I.e., how much must area $\Delta r \Delta \theta$ in (r,θ)-plane be multiplied to give its area in the (x,y)-plane?

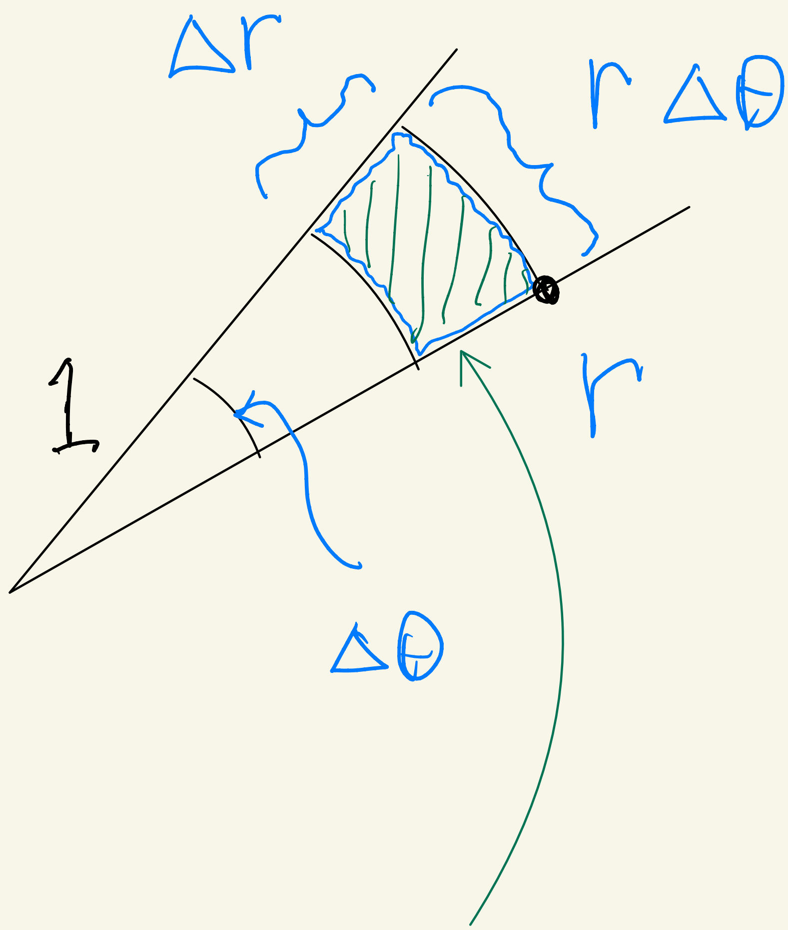
That is:

$$\Delta x \Delta y = \boxed{?} \Delta r \Delta \theta$$

Amplification factor for area

Ans: We get this from the geometry

- To get <u>amplification</u> <u>factor</u> blow up the picture —



Area in the (x,y)-plane is $\Delta A = \underbrace{r}_{\text{Amplification factor} = r}\,\Delta r\,\Delta\theta$

- **Conclude:**

$$\iint_R f(x,y)\,dA = \lim_{N\to\infty} \sum_{(x_i,y_j)\in R_{xy}} f(x_i,y_j)\,\Delta x\,\Delta y$$

$$= \underbrace{\lim_{N\to\infty} \sum_{(r_i,\theta_j)\in R_{r\theta}} f(r_i\cos\theta_j,\, r_i\sin\theta_j)\,\underbrace{r_i\,\Delta r\,\Delta\theta}_{\Delta A}}_{\text{Riemann Sum in } (r,\theta)}$$

$$= \iint_{R_{r\theta}} f(r\cos\theta,\, r\sin\theta)\, r\, dr\, d\theta$$

requires amplification factor

- Key Take Away: We are interested in evaluating an integral in (x,y)-coordinates

We express the function and draw the region in (x,y)-coords

We write the grid in (r,θ) & express a volume element in (r,θ)

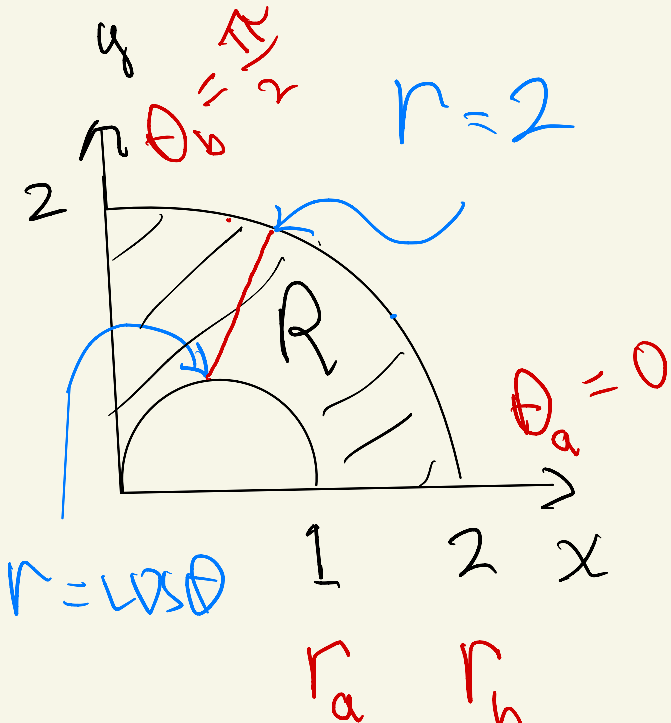
$$\Delta V_{ij} = f(r_i \cos\theta_j, r_i \sin\theta_j)\underbrace{r_i \Delta r \Delta\theta}_{\text{"}\Delta x \Delta y\text{"}}$$

$$\iint_R f(x,y)\,dA = \sum_R \Delta V_{ij} = \iint_{R_{r\theta}} \bar{f}(r,\theta)\, r\, dr\, d\theta$$

Example ① Find the mass M of a metal plate of constant density $\delta(x,y) = \delta = \text{const}$ that lies betw $r=2$, $r=\cos\theta$,

$0 \le \theta \le \pi/2$

Soln: Picture



$$\text{Mass} = \iint_R \delta \, dA$$

$$= \delta \iint_{R_{xy}} dx\,dy$$

$$= \delta \iint_{R_{r\theta}} r\,dr\,d\theta = \delta \int_0^{\pi/2}\int_{\cos\theta}^{2} r\,dr\,d\theta$$

11

$$\text{Mass} = \delta \int_0^{\pi/2} \int_{\cos\theta}^{2} r \, dr \, d\theta$$

$$= \delta \int_0^{\pi/2} \frac{r^2}{2} \Big]_{r=\cos\theta}^{r=2} d\theta = \delta \int_0^{\pi/2} \frac{2^2}{2} - \frac{\cos^2\theta}{2} \, d\theta$$

$$= 2\delta \frac{\pi}{2} - \frac{1}{2}\delta \int_0^{\pi/2} \underbrace{\cos^2\theta}_{\frac{1}{2}(1+\cos 2\theta)} \, d\theta$$

$$= \delta\pi - \frac{1}{4}\delta \int_0^{\pi/2} 1 + \cos 2\theta \, d\theta$$

$$= \delta\pi - \frac{1}{4}\delta \left(\theta + \frac{1}{2}\sin 2\theta\right)\Big]_0^{\pi/2}$$

$$= \delta\pi - \delta\frac{\pi}{8} - \frac{1}{8}\cancel{\sin 2\theta}\Big]_0^{\pi/2} = \delta \frac{7\pi}{8}$$

0

☒ Set up the integral in polar coords for radius of gyration about the x-axis:

<u>Soln</u>: Rad Gyration $= \sqrt{\dfrac{I_x}{M}}$

$$I_x = \iint_R y^2 \sigma \, dA$$

$$= \iint_{R_{r\theta}} (r\sin\theta)^2 \sigma \, r \, dr \, d\theta$$

$$= \sigma \int_0^{\pi/2} \int_{\cos\theta}^{2} r^3 \sin^2\theta \, dr \, d\theta$$

$$M = \frac{7\pi}{8}\sigma$$

✓

■ Important example –

- The "bell shaped curve" of probability theory is called the <u>Gaussian</u> <u>Distribution</u>

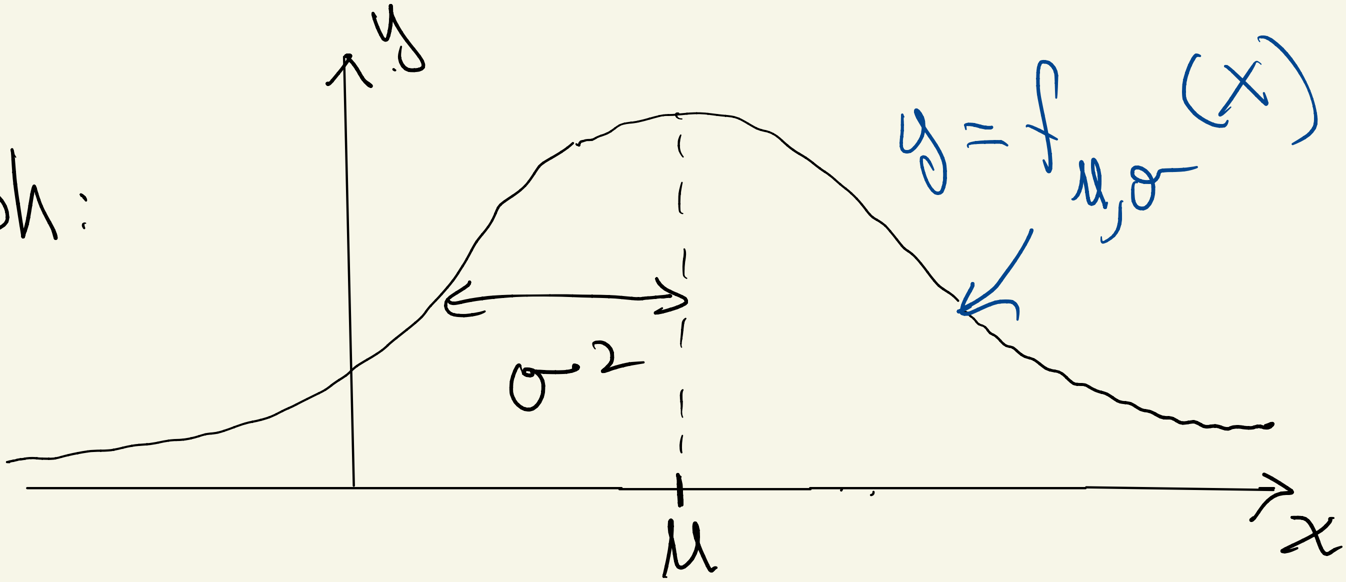
$$f(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}} = f_{\mu,\sigma}(x)$$

μ = mean

σ^2 = variance

σ = standard deviation

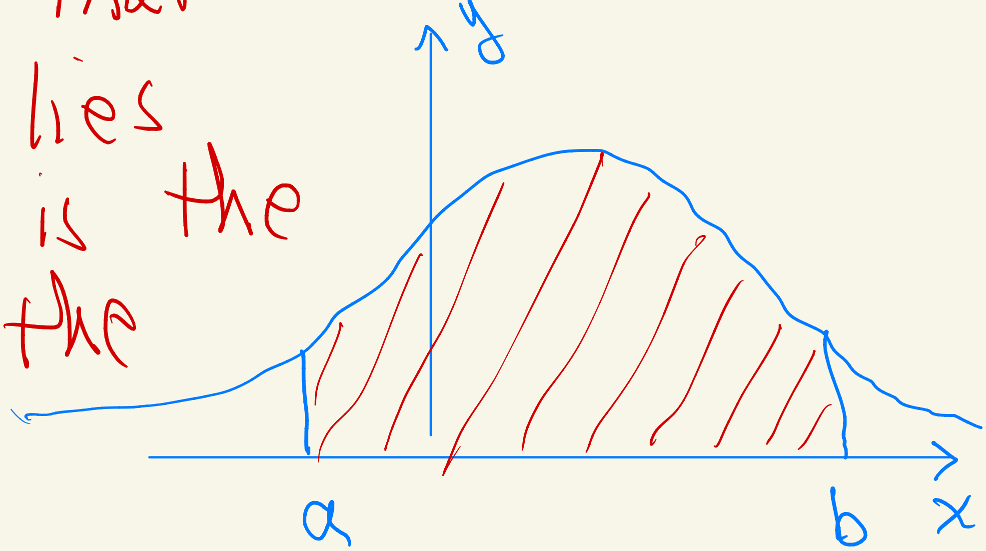
Graph:



- <u>Theorem</u>: The average of N outcomes of a random variable (appropriately rescaled) <u>always</u> tends to $f_{\mu,\sigma}$ for some μ, σ
- Background: in the modern theory of probability (Kolmogorov)

$$\text{Prob}\{x \in [a,b]\} = \int_a^b f_{\mu,\sigma}(x)\,dx$$

"The probability that the outcome lies betw a & b is the area under the graph"



15

- Probability is a number between zero and one - So for the theory to make sense, we must have

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = 1$$

Problem: Prove This!

Soln: simplify and evaluate in polar coordinates

- Change Variables

$$f_{\mu,\sigma}(x) = \frac{1}{\sqrt{2\pi}\,\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2}$$

So...

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} \frac{1}{\sigma} e^{-\left(\frac{x-\mu}{\sqrt{2}\,\sigma}\right)^2} dx$$

Set: $u = \frac{x-\mu}{\sqrt{2}\,\sigma}$

$du = \frac{dx}{\sqrt{2}\,\sigma}$

$$= \int_{-\infty}^{\infty} \frac{1}{\sqrt{\pi}} e^{-u^2} du$$

$$\int_{-\infty}^{\infty} f_{\mu,\sigma}(x)\,dx = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} e^{-u^2}\,du$$

Conclude: it suffices to show

$$\int_{-\infty}^{\infty} e^{-u^2}\,du = \sqrt{\pi}$$

or

$$\int_{0}^{\infty} e^{-u^2}\,du = \frac{\sqrt{\pi}}{2}$$

Problem - there is no Math 21B substitution that works!

For example: $v = u^2 \Rightarrow dv = 2u\,du$??

Doesn't work!

The new trick employs polar coordinates —

Set $I = \int_0^\infty \int_0^\infty e^{-x^2-y^2}\,dy\,dx$

$$= \int_0^\infty e^{-x^2} \underbrace{\left[\int_0^\infty e^{-y^2}\,dy\right]}_{\text{constant!}} dx$$

$$= \underbrace{\left[\int_0^\infty e^{-y^2}\,dy\right]}_{\text{same}} \underbrace{\int_0^\infty e^{-x^2}\,dx}_{\text{integrals}}$$

$$= \left(\int_0^\infty e^{-x^2}\,dx\right)^2 \Rightarrow \int_0^\infty e^{-x^2}\,dx = \sqrt{I}$$

Thus to evaluate:

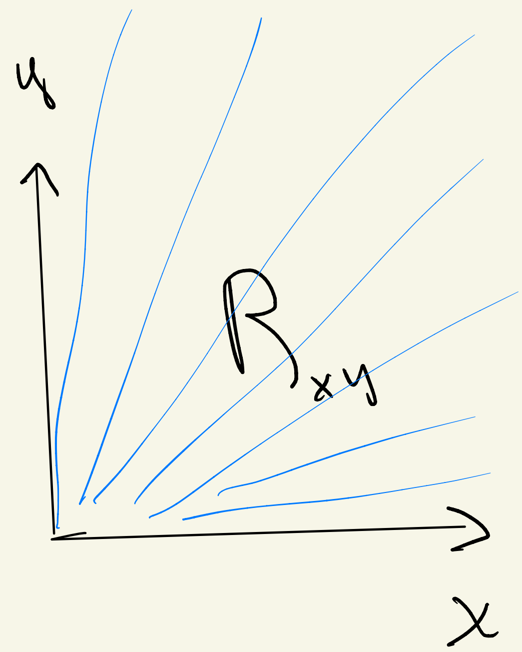
$$I = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\,dx\,dy$$

polar coordinates: $r^2 = x^2 + y^2$

$$R_{xy}: [0,\infty] \times [0,\infty]$$

$$R_{r\theta}: \quad 0 \le r \le \infty, \quad 0 \le \theta \le \pi/2$$

thus



$$I = \int_0^{\pi/2} \int_0^\infty e^{-r^2}\, r\,dr\,d\theta$$

20

$$I = \int_0^{\pi/2}\int_0^{\infty} e^{-r^2}\, r\, dr\, d\theta$$

$u = r^2 \quad du = 2r\,dr$

$$= \int_0^{\pi/2} \frac{1}{2}\int_0^{\infty} e^{-u}\, du$$

$$= \int_0^{\pi/2} \frac{1}{2}\left[-e^{-u}\right]_{u=0}^{u=\infty} d\theta$$

$$= \int_0^{\pi/2} \frac{1}{2}\, d\theta = \frac{1}{2}\theta\Big]_0^{\pi/2}$$

$$= \pi/4 \qquad \int_0^{\infty} e^{-u^2} du = \sqrt{I} = \frac{\sqrt{\pi}}{2}$$

✓